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Completion of codes with finite bi-decoding delays☆

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Abstract

Let A^* be a free monoid generated by a set A and let $X \subseteq A^*$ be a code with property P. The embedding of X into a complete code $Y \subseteq A^*$ with the same property P is called the completion of X. The method of completion of rational bifix codes and codes with finite decoding delays have been investigated by a number of authors. In this paper, we provide a general method of construction for completing the codes with finite bi-decoding delays. As a consequence, the completion method of rational bifix codes and codes with finite decoding delays is extended and applied to codes with finite bi-decoding delays.

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1. Introduction

Let A^* be a *free monoid* generated by a set A. Then, we call an element $w \in A^*$ a *word* over A and any subset of A^* a *language* over A. By a *prefix relation* on A^*, we mean the relation $u \overset{P}{=} v$, where u is a prefix of v or v is a prefix of u,

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i.e., $v = uu'$ or $u = vv'$ for some $u', v' \in A^*$. The *suffix relation* is defined analogously by $u \overset{S}{=} v$, where u is a suffix of v or v is a suffix of u, i.e., $v = u'u$ or $u = v'v$ for some $u', v' \in A^*$. Let $A^+ = A^* \backslash 1$, where 1 is the identity element of A^*. Then, we call a language $X \subseteq A^+$ a *prefix language* (respectively, a *suffix language*) if no word of X is a prefix (respectively, suffix) of the others. A language $X \subseteq A^+$ is called a *code* if it is a base of a free submonoid of A^*. In this connection, a prefix language (respectively, suffix language) is a code and we call it a *prefix code* (respectively, *suffix code*). A language which is both a prefix code and a suffix code is called a *bifix code*. As a generalization of the prefix codes, we consider *codes with finite decoding delay*. A language $X \subseteq A^+$ is said to have finite decoding delay from left to right if there exists an integer d satisfying the following condition:

$$\forall x, x' \in X, \ \ \forall y \in X^d, \ \ \forall u \in A^* : xyu \in x'X^* \Rightarrow x = x'.$$

The smallest integer satisfying the above condition is called the decoding delay (from left to right) of X. As a language with finite decoding delay from left to right is a code and hence the prefix codes are just the codes with decoding delay 0. Dually, we also have codes with decoding delay from right to left. Although codes with finite decoding delay from left to right (respectively, from right to left) is a generalization of prefix codes (respectively, suffix codes), it seems that there is no correspondence relation, regarded as the generalized prefix relations (respectively, suffix relations), to describe the codes with finite decoding delays. Recently, these generalized relations have been considered by the authors in [8] and they called these generalized relations the *G-prefix relations* (respectively, *G-suffix relations*), where G is a language over A, i.e., $G \subseteq A^*$. The G-prefix relation "$\overset{GP}{=}$" on A^* determined by $G \subseteq A^*$ is defined by

$$u \overset{GP}{=} v \ \ \Leftrightarrow \ \ u \in vG \quad \text{or} \quad v \in uG, \quad \forall u, \ v \in A^*.$$

Dually, the G-suffix relation "$\overset{GS}{=}$" on A^* is defined by

$$u \overset{GS}{=} v \ \ \Leftrightarrow \ \ u \in Gv \quad \text{or} \quad v \in Gu, \quad \forall u, \ v \in A^*.$$

We now denote $u \overset{GP}{\leqslant} v$ if $v \in uG$ and call u a G-left factor of v. Likewisely, we define $u \overset{GS}{\leqslant} v$ and call u the G-right factor of v. It is clear that the A^*-prefix relations and the A^*-suffix relations are the ordinal prefix relations and the ordinal suffix relations, respectively. For the sake of brevity, we just denote $u \overset{A^*P}{\leqslant} v$ ($u \overset{A^*S}{\leqslant} v$) by $u \overset{P}{\leqslant} v$ ($u \overset{S}{\leqslant} v$). Clearly, $\overset{P}{\leqslant}$ and $\overset{S}{\leqslant}$ are both orders on A^* and we call them the *prefix order* and the *suffix order*, respectively. As the corresponding sets of the prefix sets, we define the *G-prefix sets*. We call the set $X \subseteq A^+$ a G-prefix set for $G \subseteq A^*$ if no two different words of X have the G-prefix relation. However, for an arbitrary fixed language $G \subseteq A^*$, a G-prefix set is not necessarily a code. It was shown by the authors in [8] that for a given subset $X \subseteq A^+$, if X is a $(X^dA^*)(X^dA^*)^{-1}$-prefix set then X must be a code. These codes are just the codes with finite decoding delays from left to right. The codes with finite decoding delay from right to left can be dually described by using G-suffix relations. Thus, we may discuss the codes with finite decoding delay (from left to right and also from right to left) by using the G-prefix relations as well as the G-suffix relations. Later on, we will further discuss the codes with finite bi-decoding delay, i.e., those codes with finite decoding delay both from left to right and from right to left. We henceforth call a code with decoding delay d from left to right a *d-prefix code*; a code with decoding delay d' from right to left a *d'-suffix code* and a code with bi-decoding delay (d,d') (with decoding delay d from left to right and with decoding delay d' from right to left) a *(d,d')-bifix code*.

Given a code $X \subseteq A^+$ with the property P, one would naturally ask: Is there any procedure for embedding a code X into a complete code $Y \subseteq A^+$ which still shares the same property P? This question is an interesting question in the theory of codes and we call it the completion of X. The completion of prefix codes is not difficult; however, if we proceed the completion problem from two conversed directions along the prefix codes, then we can immediately see that the situation is rather complicated. On one hand, from the strengtheness point of view, there are bifix codes and on the other hand, from the generalization point of view, there are d-prefix codes. The completion problem for these codes has already achieved considerable progress in the last two decades. In particular, Perrin in 1982 has obtained a construction method for embedding a finite bifix code into a rational complete bifix code [5] and in 1995, Zhang and Shen have extended Perrin's result and in fact, they have obtained a construction method for embedding a rational bifix code into a rational complete bifix code [7]. Recently, Bruyere and Perrin have further simplified this method by considering the maximal bifix codes [3]. Concerning the d-prefix codes, Schutzenberger has already shown in 1966 that every finite maximal code with finite decoding delay is a prefix code [6]. In this connection, Bruyere et al. in 1990 obtained a construction method for embedding a rational (respectively, thin) d-prefix code into a complete rational (respectively, thin) d-prefix code [4]. Since we have successfully obtained the construction methods of completion for the prefix codes and its generalized d-prefix codes, and also the bifix codes, one would naturally ask whether we can find a completion method for the (d,d')-bifix codes which is a generalization of bifix codes? This problem was in fact proposed by Bruyere and Latteux in 1996 [2]; however, the method on d-prefix codes given in [4] indicated that there does not exist any no relationship between the prefix codes and d-prefix codes. Consequently, we are unable to find an easy access from the bifix codes to the (d,d')-bifix codes. Nevertheless, we now observe that the recent method for d-prefix codes proposed by the authors in [8] can be adopted to link up the prefix codes because we can find a natural extension of the completion of prefix codes by using the G-prefix relations. In this paper, we will adopt the generalized method from the prefix codes to the d-prefix codes to deal with the (d,d')-bifix codes via the bifix codes. A completion method for codes with finite bi-decoding delays is obtained. The reader is referred to Berstel and Perrin [1] for terminology and notations not given in this paper.

2. (*d*, *d′*)-bifix codes

In this section, we concentrate on (d,d')-bifix codes. We first introduce some basic notations. For a language $L \subseteq A^*$, we let

$$LA^- = L(A^+)^{-1}, \quad A^-L = (A^+)^{-1}L$$

be the sets of proper prefixes and proper suffixes of words in L, respectively. The following symbols

$$F(L), P(L), S(L)$$

are used to denote the sets of all minimal elements of L for factor order, prefix order and suffix order, respectively.

We first cite some important results that have been recently obtained by the authors in [8].

Definition 2.1. Let $H, G \subseteq A^*$. If a subset $H' \subseteq H$ satisfies the following conditions:
(i) H' is a G-prefix set;
(ii) for all $h \in H$, there exists $h' \in H' : h' \overset{GP}{=} h$.
Then H' is called a *G-prefix kernel* of H.

In addition, if condition (ii) is being replaced by the following condition:
(ii)′ for all $h \in H$, there exists $h' \in H' : h' \overset{GP}{\leqslant} h$.
Then H' is called a *G-prefix root* of H and is denoted by $P_G(H)$.

The G-suffix kernel and G-suffix root can be defined dually. Clearly, we can easily observe that $P_{A^*}(L) = P(L)$ and $S_{A^*}(L) = S(L)$.

By a *thin language* $L \subseteq A^*$, we mean the language L satisfies the condition $(A^*)^{-1}L(A^*)^{-1} \neq A^*$. Thus, a code $X \subseteq A^+$ is thin if the code X is a thin language over A (see [1]).

Let $X \subseteq A^+$ be a d-prefix code and let

$$\check{X}_L = X^dA^* - (X^{d+1}A^- \cup X^{d+1}A^*),$$
$$U_{XL} = (X^dA^*)(X^dA^*)^{-1}.$$

Then, we have the following results (see [8]).

Theorem 2.1. (See Zhang et al. [8, Propositions 4.1–4.3]):
(1) *$X' \subseteq \check{X}_L$ is a U_{XL}-prefix set $\Rightarrow Y = X \cup X'$ is a d-prefix code.*
(2) *X' is a U_{XL}-prefix kernel of $\check{X}_L \Rightarrow Y = X \cup X'$ is a complete d-prefix code.*
(3) *If $Y = X \cup X'$ is a maximal d-prefix code* (*i.e.*, *maximal in the family of d-prefix codes over A*), *where $X' \subseteq \check{X}_L$, then X' is a U_{XL}-prefix kernel of $\check{X}L$.*
(4) *If X and $X' \subseteq \check{X}_L$ are both thin, then $Y = X \cup X'$ is a maximal d-prefix code if and only if X' is a U_{XL}-prefix kernel of $\check{X}_L$.*

Our aim is to extend the above results from d-prefix codes to (d,d')-bifix codes. For a (d,d')-bifix code $X \subseteq A^+$, we extend the above two sets $\check{X}_L$ and U_{XL} to the following five sets which are related to the code X:

$$\check{X} = (X^d A^* \cap A^* X^{d'}) - (X^{d+1}A^- \cup A^- X^{d'+1} \cup X^{d+1}A^* \cup A^* X^{d'+1}),$$
$$\check{X}_L = X^d A^* - (X^{d+1}A^- \cup X^{d+1}A^*),$$
$$\check{X}_R = A^* X^{d'} - (A^- X^{d'+1} \cup A^* X^{d'+1}),$$
$$U_{XL} = (X^d A^*)(X^d A^*)^{-1},$$
$$U_{XR} = (A^* X^{d'})^{-1}(A^* X^{d'}).$$

It is easy to see that $\check{X}_L \cap \check{X}_R = \check{X}$ and if $y_1 \in \check{X}_L$, $y_2 \in \check{X}_R$, then $y_1 y_2 \in \check{X}$, i.e., $\check{X}_L \check{X}_R \subseteq \check{X}$.

We start with the following propositions.

Proposition 2.2. *Let $X \subseteq A^+$ be a (d,d')-bifix code. If $X' \subseteq \check{X}$ is both a U_{XL}-prefix set and a U_{XR}-suffix set, then $Y = X \cup X'$ is a (d,d')-bifix code.*

Proof. Since $\check{X} = \check{X}_L \cap \check{X}_R$, X' is both a U_{XL}-prefix set contained in $\check{X}_L$ and a U_{XR}-suffix set contained in $\check{X}_R$. Now, by Theorem 2.1(1) and its dual, it is known that Y is both a d-prefix code and a d'-suffix code. The conclusion is therefore proved immediately. $\square$

Proposition 2.3. *Let $X \subseteq A^+$ be a (d,d')-bifix code. Suppose that $K \subseteq \check{X} \neq \emptyset$. If K is a U_{XL}-prefix kernel (U_{XR}-suffix kernel) of $\check{X}$, then K is a U_{XL}-prefix kernel (U_{XR}-suffix kernel) of $\check{X}_L(\check{X}_R)$.*

Proof. Let K be a U_{XL}-prefix kernel of $\check{X}$. Since $\check{X} \neq \emptyset$ and $\check{X} = \check{X}_L \cap \check{X}_R$, it follows that $\check{X}_L \neq \emptyset$ and $\check{X}_R \neq \emptyset$. For any $y_L \in \check{X}_L$, we claim that there exists an element $y_0 \in K$ such that $y_L \overset{U_{XL}P}{=} y_0$. In proving our claim, we first pick a word $y_R \in \check{X}_R$ and a word $x \in X^d$. We now show that $y_L x y_R \in \check{X}$. In fact, we have $y_L x y_R \in X^d A^* \cap A^* X^{d'}$. If $y_L x y_R \in X^{d+1}A^- \cup X^{d+1}A^*$ or $y_L x y_R \in A^- X^{d'+1} \cup A^* X^{d'+1}$, then we have $y_L \in X^{d+1}A^- \cup X^{d+1}A^*$ or $y_R \in A^- X^{d'+1} \cup A^* X^{d'+1}$, respectively. These cases contradict to the facts $y_L \in \check{X}_L$ and $y_R \in \check{X}_R$. Hence, $y_L x y_R \notin X^{d+1}A^- \cup X^{d+1}A^* \cup A^- X^{d'+1} \cup A^* X^{d'+1}$. This shows that $y_L x y_R \in \check{X}$. Since K is a U_{XL}-prefix kernel of $\check{X}$, there exists a $y_0 \in K$ such that $y_0 \overset{U_{XL}P}{=} y_L x y_R$. If $y_0 \overset{U_{XL}P}{\geqslant} y_L x y_R$, then we have $y_0 \in y_L X^d A^*$ and consequently, $y_0 \overset{U_{XL}P}{\geqslant} y_L$. If $y_0 \overset{U_{XL}P}{\leqslant} y_L x y_R$, then $y_L x y_R = y_0 u$, where $u \in U_{XL}$. Because $u \in U_{XL}$, we know that there exists $v \in A^*$ such that $uv \in X^d A^*$. Thereby, we have $y_L(x y_R v) = y_0(uv)$, where $x y_R v$, $uv \in X^d A^*$. This leads to $y_L \overset{U_{XL}P}{=} y_0$ and our claim is established. In other words, K is a U_{XL}-prefix kernel of $\check{X}_L$. Similarly, we can also prove that K is a U_{XR}-suffix kernel of $\check{X}_R$ if K is a U_{XR}-suffix kernel of $\check{X}$. $\square$

Proposition 2.4. *Let $X \subseteq A^+$ be a (d,d')-bifix code. Then the following statements hold*:

(1) *If X is a maximal (d,d')-bifix code, then $\check{X} = \emptyset$.*

(2) *If $\check{X} = \emptyset$, then X is a complete code.*

Proof. (1) Let X be a maximal (d,d')-bifix code. If $\check{X}\neq\emptyset$, then for any $s\in\check{X}$, by Proposition 2.2, we know that $X\cup\{s\}$ is a (d,d')-bifix code. Obviously, this contradicts our assumption and hence $\check{X}=\emptyset$.

(2) Suppose that $\check{X}=\emptyset$. Since $\check{X}_L\check{X}_R\subseteq\check{X}$, we see that $\check{X}_L=\emptyset$ or $\check{X}_R=\emptyset$. Assume that $\check{X}_L=\emptyset$. Then, clearly, for any word $w\in A^*$, either there exists $u\in X^*$ such that $uw\in X^dA^*-X^{d+1}A^*$, or there exists $v\in X^*$ such that $v^{-1}w\in X^dA^*-X^{d+1}A^*$. Since $\check{X}_L=\emptyset$, we have $uw\in X^{d+1}A^-$ or $v^{-1}w\in X^{d+1}A^-$. This shows that w is a factor of some words in X^*. On the other hand, if $\check{X}_R=\emptyset$, then we can prove the similar result. Therefore X is a complete code. □

By using the above proposition, we immediately deduce the following result:

Proposition 2.5. *Let $X\subseteq A^+$ be a thin (d,d')-bifix code. Then the following statements are equivalent*:
(1) *X is a maximal d-prefix code.*
(2) *X is a maximal (d,d')-bifix code.*
(3) *X is a maximal d'-suffix code.*

The following theorem provides a method for the completion of a (d,d')-bifix code.

Theorem 2.6. *Let $X\subseteq A^+$ be a (d,d')-bifix code and $X'\subseteq\check{X}$. Then the following statements hold*:
(1) *If X' is both a U_{XL}-prefix kernel of $\check{X}$ and a U_{XR}-suffix set* (*respectively*, *X' is both a U_{XR}-suffix kernel of $\check{X}$ and a U_{XL}-prefix set*), *then $Y=X\cup X'$ is a complete (d,d')-bifix code.*
(2) *If $Y=X\cup X'$ is a maximal (d,d')-bifix code, then X' is both a U_{XL}-prefix kernel of $\check{X}$ and a U_{XR}-suffix set or X' is both a U_{XR}-suffix kernel of $\check{X}$ and a U_{XL}-prefix set.*

Proof. (1) If $\check{X}=\emptyset$, then by Proposition 2.4, X is a complete code and the conclusion holds. Now we suppose that $\check{X}\neq\emptyset$. Let X' be a U_{XL}-prefix kernel of $\check{X}$. Then by Proposition 2.3, X' is also a U_{XL}-prefix kernel of $\check{X}_L$. Moreover, by Theorem 2.1(2), we know that $Y=X\cup X'$ is a complete d-prefix code. Since X' is both a U_{XL}-prefix set and a U_{XR}-suffix set, by Proposition 2.2, Y is a (d,d')-bifix code. Therefore, Y is a complete (d,d')-bifix code. For the case that X' is a U_{XR}-suffix kernel of $\check{X}$ and a U_{XL}-prefix set, the proof is similar and is hence omitted.

(2) Let $Y=X\cup X'$ be a maximal (d,d')-bifix code. Then, it is clear that X' is both a U_{XL}-prefix set and a U_{XR}-suffix set. If X' is not a U_{XL}-prefix kernel of $\check{X}$ and is not a U_{XR}-suffix kernel of $\check{X}$, then there exist $y_1,y_2\in\check{X}-X'$ such that $X'\cup\{y_1\}$ and $X'\cup\{y_2\}$ are a U_{XL}-prefix set and a U_{XR}-suffix set, respectively. Clearly, $y_1y_2\in\check{X}-X'$. We claim that $X'\cup\{y_1y_2\}$ is both a U_{XL}-prefix set and a U_{XR}-suffix set. In fact, if there is a $y_0\in X'$ such that $y_0\overset{U_{XL}}{=}y_1y_2$, then since y_2 has a left factor in X^d, we know that $y_0\overset{U_{XL}}{=}y_1$. This contradicts to our choice of y_1 because y_1 is a word such that $X'\cup\{y_1\}$

is a U_{XL}-prefix set. Hence, $X' \cup \{y_1 y_2\}$ is a U_{XL}-prefix set. Similarly, $X' \cup \{y_1 y_2\}$ is a U_{XR}-suffix set. Thus, by Proposition 2.2, $X \cup X' \cup \{y_1 y_2\}$ is a (d,d')-bifix code. This contradicts to our assumption that $Y = X \cup X'$ is a maximal (d,d')-bifix code. Therefore, X' is either a U_{XL}-prefix kernel of $\check{X}$ or a U_{XR}-suffix kernel of $\check{X}$. □

It is well known that if $X \subseteq A^+$ is a thin code, then X is a maximal code if and only if X is a complete code [1]. Thus, for the thin (d,d')-bifix codes, we obtain a stronger version of Theorem 2.6.

Theorem 2.7. *Let $X \subseteq A^+$ be a thin (d,d')-bifix code. Suppose that $X' \subseteq \check{X}$ is both a thin U_{XL}-prefix set and a thin U_{XR}-suffix set. Then we have the following characterizations for the U_{XL}-prefix kernels and the U_{XR}-suffix kernels of $\check{X}$.*

(1) *$Y = X \cup X'$ is a maximal (d,d')-bifix code if and only if X' is a U_{XL}-prefix kernel of $\check{X}$ or a U_{XR}-suffix kernel of $\check{X}$.*

(2) *X' is a U_{XL}-prefix kernel of $\check{X}$ if and only if X' is a U_{XR}-suffix kernel of $\check{X}$.*

Proof. (1) Since a thin and complete code must be a maximal code, the conclusion follows immediately from Theorem 2.6.

(2) Let X' be a U_{XL}-prefix kernel of $\check{X}$. Then by (1) of this Theorem, $X \cup X'$ is a maximal (d,d')-bifix code. By Proposition 2.5, $X \cup X'$ is a maximal d'-suffix code. By the dual of Theorem 2.1(4), $X \cup X'$ is a U_{XR}-suffix kernel of $\check{X}_R$. Since $\check{X} \subseteq \check{X}_R$, X' must be a U_{XR}-suffix kernel of $\check{X}$. The proof of the converse statement is similar and is hence omitted. □

By Theorem 2.7, we see that for embedding a thin (d,d')-bifix code X into a maximal code, we only need to find a thin language which is both a U_{XL}-prefix kernel and a U_{XR}-suffix kernel of $\check{X}$.

In the following, we are going to link up two bifix codes by a (d,d')-bifix code X and then we will find a set which is both a U_{XL}-prefix kernel and a U_{XR}-suffix kernel of $\check{X}$. In fact, for a given (d,d')-bifix code $X \subseteq A^+$, we can always form the following two sets which are related to $X \subseteq A^+$.

$$E_X = S(X^{d'})XP(X^d),$$
$$D_X = S(X^{d'})P(X^d).$$

By the following Proposition 2.8, we can easily see that the above languages E_X and D_X are both bifix codes. In fact, these codes play a crucial role in finding a subset of $\check{X}$ which is both a U_{XL}-prefix kernel and U_{XR}-suffix kernel of $\check{X}$.

Proposition 2.8. *Let $X \subseteq A^+$ be a (d,d')-bifix code. Then, for any $k \geqslant 0$, $S(X^{d'})X^kP(X^d)$ are bifix codes.*

Proof. Let $x, y \in S(X^{d'})X^kP(X^d)$. Then we can write $x = x_1x_2x_3$ and $y = y_1y_2y_3$, where $x_1, y_1 \in S(X^{d'})$, $x_2, y_2 \in X^k$, $x_3, y_3 \in P(X^d)$. If $x = yu$, i.e., $x_1x_2x_3 = y_1y_2y_3u$, then we have $x_1x_2 = y_1y_2$ since X is a d-prefix code. Consequently, $x_1 = y_1$, $x_2 = y_2$ since X is a code. Moreover, $x_3 = y_3$ since $P(X^d)$ is a prefix code. This proves that $x = y$. Analogously, we can also show that $y = xu$ implies that $x = y$. Therefore, $S(X^{d'})X^kP(X^d)$ must be a bifix code. □

By Proposition 2.8, we immediately see that the languages E_X and D_X are bifix codes.

Proposition 2.9. *Let $X \subseteq A^+$ be a (d,d')-bifix code. Suppose that $W \subseteq X^dA^* \cap A^*X^{d'}$. Then the following properties hold*:
(1) *W is a U_{XL}-prefix set if and only if $S(X^{d'})WP(X^d)$ is a prefix code.*
(2) *W is a U_{XR}-suffix set if and only if $S(X^{d'})WP(X^d)$ is a suffix code.*
(3) *W is both a U_{XL}-prefix set and a U_{XR}-suffix set if and only if $S(X^{d'})WP(X^d)$ is a bifix code.*

Proof. We only need to prove (1) since the proof of (2) is symmetric to (1), also (3) is just a combination of (1) and (2). Assume that $S(X^{d'})WP(X^d)$ is a prefix code. If W is not a U_{XL}-prefix set, then there exist $x, y \in W$ such that $x = yu$, where $u \in U_{XL} \backslash 1$. Recall that $U_{XL} = (X^dA^*)(X^dA^*)^{-1}$, we have $x_1, y_1 \in P(X^d)$ and $u_1 \in A^+$ such that $xx_1 = yy_1u_1$ $xx_1u_1 = yy_1$ or $xx_1 = yy_1$. The first two cases contradict to the prefixity of $S(X^{d'})WP(X^d)$ and the last one implies that $x = y$ since X is a code and $x_1, y_1 \in X^d$; this contradicts to the assumption $u \neq 1$. Hence W is a U_{XL}-prefix set. Conversely, assume that W is a U_{XL}-prefix set. If $S(X^{d'})WP(X^d)$ is not a prefix code, then we have $x_1x_2x_3, y_1y_2y_3 \in S(X^{d'})WP(X^d)$, where $x_1, y_1 \in S(X^{d'}), x_2, y_2 \in W, x_3, y_3 \in P(X^d)$ such that $x_1x_2x_3 = y_1y_2y_3u$ with $u \in A^+$. Since $W \subseteq X^dA^* \cap A^*X^{d'}$ and X is a d-prefix code, we have $x_1 = y_1$ and clearly $x_2x_3 = y_2y_3u$. This shows that x_2, y_2 are related by the U_{XL}-prefix relation. However, this contradicts to our assumption that W is a U_{XL}-prefix set. Hence $S(X^{d'})WP(X^d)$ must be a prefix code. □

Let $G \subseteq A^+$. Then we call a code $X \subseteq G$ with property P maximal in G if for any word $g \in G - X$, $X \cup \{g\}$ is not a code with property P.

The following theorem, using the bifix code E_X, describes a method to find a language W which is both a U_{XL}-prefix kernel of $\check{X}$ and a U_{XR}-suffix kernel of $\check{X}$. For the sake of brevity, we denote $S(X^{d'})\check{X}P(X^d)$ by B_X. Clearly, we have

$$B_X = (D_XA^* \cap A^*D_X) - (S(X^{d'})X^{d+1}A^* \cup S(X^{d'})X^{d+1}A^- \cup A^*X^{d'+1}P(X^d) \cup A^-X^{d'+1}P(X^d).$$

Theorem 2.10. *Let $X \subseteq A^+$ be a thin (d,d')-bifix code. Suppose that $W \subseteq X^dA^* \cap A^*X^{d'}$ is also thin. Then $S(X^{d'})WP(X^d)$ is a maximal bifix code in B_X if and only if W is both a U_{XL}-prefix kernel and a U_{XR}-suffix kernel of $\check{X}$.*

Proof. Let $S(X^{d'})WP(X^d)$ be a maximal bifix code in B_X. Then by Proposition 2.9, we know that W is both a U_{XL}-prefix set and a U_{XR}-suffix set. If W is not a U_{XL}-kernel of $\check{X}$, then by Theorem 2.7, W is not a U_{XR}-suffix kernel of $\check{X}$. Thus, there exist $y', y'' \in \check{X} - W$ such that $W \cup \{y'\}$ is a U_{XL}-prefix set and $W \cup \{y''\}$ is a U_{XR}-suffix set. This leads to

$W \cup \{y'y''\}$ both a U_{XL}-prefix set and a U_{XR}-suffix set. It is clear that $y'y'' \in \check{X}$. Again by Proposition 2.9, we immediately see that $S(X^{d'})(W \cup \{y'y''\})P(X^d)$ is a bifix code. This contradicts our assumption that $S(X^{d'})WP(X^d)$ is a maximal bifix code in B_X. Therefore, W is both a U_{XL}-kernel of $\check{X}$ and a U_{XR}-kernel of $\check{X}$.

Conversely, let W be both a U_{XL}-prefix kernel and a U_{XR}-suffix kernel of $\check{X}$. Then, by Proposition 2.9, $S(X^{d'})WP(X^d)$ is a bifix code. We claim that, for any word $swp \in B_X - S(X^{d'})WP(X^d)$, where $s \in S(X^{d'})$, $w \in \check{X}$, $p \in P(X^d)$, the set $S(X^{d'})WP(X^d) \cup \{swp\}$ must not be a bifix code. If this is not true, then by Proposition 2.9 again, we know that $W \cup \{w\}$ is a U_{XL}-prefix set. This contradicts the fact that W is a U_{XL}-prefix kernel of $\check{X}$. This finishes the proof. □

Theorem 2.11. *Let $X \subseteq A^+$ be a thin (d,d')-bifix code and a thin language $W \subseteq \check{X}$ such that $S(X^{d'})WP(X^d)$ is a maximal bifix code in B_X. Then $X \cup W$ is a maximal (d,d')-bifix code over A.*

Proof. By Theorem 2.10, W is a U_{XL}-prefix kernel of $\check{X}$ and a U_{XR}-suffix kernel of $\check{X}$. Furthermore, by Theorem 2.7, $X \cup W$ is a maximal (d,d')-bifix code over A. □

3. Completion of rational (d, d')-bifix codes

For the completion of rational bifix codes, there are two kind of algorithms which have been fully described in [3,7], respectively. Let $X \subseteq A^+$ be a rational (d,d')-bifix code. Then we have the rational bifix codes D_X, E_X and the set B_X. By the method given in [3] or [7], we can construct a rational bifix codes $\bar{D}_X$ such that $\bar{D}_X \cup D_X$ is a maximal bifix code with $D_X \cap \bar{D}_X = \emptyset$. In the following proposition, we shall prove that $E_X \cup \bar{D}_X$ is a bifix code. By using similar method as above, we can also construct a bifix code Y such that $Y \cup E_X \cup \bar{D}_X$ is a maximal bifix code with $Y \cap (E_X \cup \bar{D}_X) = \emptyset$. In this section, the notations X, D_X, E_X, $\bar{D}_X$ and Y always indicate such codes. For $L_1 \subseteq A^*$ and $L_2 \subseteq A^*$, we say that L_1 is right-complete in L_2 if any word $w \in L_2$ is comparable with a word of L_1 in the prefix order.

The following lemmas describe the relation between the codes D_X, E_X, $\bar{D}_X$ and B_X.

Lemma 3.1. *Among the codes $E_X, \bar{D}_X$ and B_X, they have no prefix relations and suffix relations between each other. Moreover, the code $E_X \cup \bar{D}_X \cup B_X$ is both right-complete and left-complete in A^*.*

Proof. Since $D_X \cup \bar{D}_X$ is a bifix code, each word in $\bar{D}_X$ has no prefixes in D_X and is also not a prefix of D_X. However, each word in $E_X \cup B_X$ has a prefix in D_X. Hence, there do not exist any prefix relations between $E_X \cup B_X$ and $\bar{D}_X$. Moreover, each word in E_X is a prefix of $S(X^{d'})X^{d+1}$. By the construction of B_X, we can see that each word in B_X has no prefixes in $S(X^{d'})X^{d+1}$ and is not a prefix of $S(X^{d'})X^{d+1}$. Thus, there does not exist any prefix relations between E_X and B_X. For suffix relations, the proof is similar. To see the second conclusion, we assume that $x \in A^*$ has no prefix relation with $\bar{D}_X$. Then, because $D_X \cup \bar{D}_X$ is a maximal prefix code, x has a prefix relation with D_X. If $x \overset{P}{\leqslant} y$ for $y \in D_X$, then x is clearly a prefix of E_X. If $x \overset{P}{>} y$, then $x = uv$, where $u \in S(X^{d'})$, $v \in P(X^d)A^+$. Therefore, two cases arises: either $v \in P(X^d)(XA^+ \cup XA^-)$ or $v \in P(X^d)(A^+ - (XA^+ \cup XA^-))$. For the former case, we see that x have prefix relation with E_X, and for the latter case, we have $v \in \check{X}_L$. Since $\check{X}_L\check{X}_R \subseteq \check{X}$, v is a prefix of $\check{X}$ and consequently, x is a prefix of B_X. Therefore, $E_X \cup \bar{D}_X \cup B_X$ is right-complete in A^*. Similarly, we can prove that $E_X \cup \bar{D}_X \cup B_X$ is left-complete in A^*. The proof is completed. □

Lemma 3.2. *Y is a maximal bifix code in B_X.*

Proof. Clearly, $Y \subseteq B_X$. If there exists $w \in B_X - Y$ such that $\{w\} \cup Y$ is a bifix code. Then, by Lemma 3.1, $\{w\} \cup Y \cup E_X \cup \bar{D}_X$ is still a bifix code. This clearly contradicts our assumption that $Y \cup E_X \cup \bar{D}_X$ is maximal bifix code. Hence, Y must be a maximal bifix code in B_X. □

Lemma 3.3. *There exists a bifix code $G_X \subseteq A^* - \bar{D}_X$ satisfying the following conditions*:
(1) *$G_X \cup \bar{D}_X$ is a maximal bifix code*;
(2) $G_X = S(X^{d'})((S(X^{d'}))^{-1} G_X (P(X^d))^{-1})P(X^d)$;
(3) *E_X has no prefix in G_X*;
(4) *E_X has no suffix in G_X.*

Proof. Let $G_0 = D_X$. Then, G_0 clearly satisfies conditions (1) and (2). We now consider the code $G_1 = D_X(\bar{D}_X)^* D_X$ that satisfies conditions (1) and (2). Write $D_X(\bar{D}_X)^*D_X = D_{11} \cup D_{12}$, where $D_{11} \cap D_{12} = \emptyset$, $D_{11} \cap E_X A^- = \emptyset$ and $D_{12} \subseteq E_X A^-$. Let $G_2 = D_{11} \cup D_{12}(D_{11} \cup \bar{D}_X)^* D_{12}$. Then, we can see that G_2 still satisfies conditions (1) and (2). If G_2 does not satisfy condition (3), then we construct G_3 by using the same method. In general, if we let $G_i = D_{i1} \cup D_{i2}$, where $D_{i1} \cap D_{i2} = \emptyset$ and $D_{i1} \cap E_X A^- = \emptyset$, $D_{i2} \subseteq E_X A^-$, then we obtain that $G_{i+1} = D_{i1} \cup D_{i2}(D_{i1} \cup \bar{D}_X)^* D_{i2}$. We claim that there are at most σ steps, i.e., G_σ must satisfy condition (3), where σ is strictly greater than the lengths of the chains in $E_X A^-$ under the suffix order because E_X is rational. Clearly the lengths of the chains in $E_X A^-$ under the suffix order and the length of the chains in $A^- E_X$ under the prefix order are all finite because there chains are bounded [3]). If G_σ does not satisfy condition (3), then there exists a word $g_\sigma \in G_\sigma$ such that $g_\sigma \in E_X A^-$. Observe that $D_{i2}(D_{i1} \cup \bar{D}_X)^* D_{i2} \subseteq A^+ D_{i2}$, there is a chain $g_1 < g_2 < \cdots < g_\sigma$ under the suffix order, where $g_i \in D_{i2}$, i.e., $g_i \in E_X A^-$. However, this is clearly impossible. Hence our claim holds and we have proved that G_σ satisfies conditions (1)–(3). By using the symmetrical arguments, we can prove that a bifix code G_X also satisfies all conditions of this lemma. □

Hereafter, we always use the notation G_X to indicate the bifix code that satisfies all the given conditions (1)–(4) in Lemma 3.3.

Recall that a *positive Bernoulli distribution* π on A^* is a morphism from A^* into the multiplicative monoid of nonnegative real numbers satisfying $\sum_{a \in A} \pi(a) = 1$ and

$\pi(a)>0$ for all $a\in A$. For $L\subseteq A^*$, the value $\pi(L)=\sum_{l\in L}\pi(l)$ is called the *measure* of the language L relative to π. For the thin code $Z\subseteq A^+$, it has been proved by Berstel and Perrin in [1] that Z is a maximal code if and only if $\pi(Z)=1$ for any one positive Bernoulli distribution π on A^*.

Let π be a positive Bernoulli distribution on A^*. Then, we denote $\pi(E_X)=\alpha$ and $\pi(G_X)=\beta$. We can easily observe that $\pi(\bar{D}_X)=1-\beta$, $\pi(Y)=\beta-\alpha$. We divide G_X into four parts, namely

$$\begin{aligned}
H_1 &= G_X \cap (E_XA^* \cap A^*E_X),\\
H_2 &= G_X - (E_XA^* \cap A^*E_X),\\
H_L &= (G_X \cap A^*E_X) - E_XA^*,\\
H_R &= (G_X \cap E_XA^*) - A^*E_X.
\end{aligned}$$

Now, we denote $\pi(H_1)=m$, $\pi(H_2)=m'$. By considering the above four parts, H_1-H_R, we have the following results:

Lemma 3.4. *$H_1\cup H_R\cup\bar{D}_X\cup Y$ and $H_1\cup H_L\cup\bar{D}_X\cup Y$ are the maximal prefix codes and the maximal suffix codes, respectively. Moreover,* $\pi(H_L)=\pi(H_R)=\alpha-m=(\beta-m-m')/2$.

Proof. Since every word in $H_1\cup H_R$ has a prefix in E_X, we can easily see that $H_1\cup H_R\cup\bar{D}_X\cup Y$ is a prefix code. Now, we observe that $G_X\cup\bar{D}_X$ is a maximal bifix code, and G_X is right-complete in E_XA^*. Because there is no words in G_X which is a prefix of E_X, we can see immediately that there are no prefix relations between $H_2\cup H_L$ (they are the parts of G_X) and E_X. Consequently, the other parts of G_X, namely $H_1\cup H_R$ must be right-complete in E_XA^*. Moreover, since $\bar{D}_X\cup Y\cup E_X$ is a maximal bifix code, $\bar{D}_X\cup Y$ is right-complete in $A^*-E_XA^*-E_XA^-$. This shows that $H_1\cup H_R\cup\bar{D}_X\cup Y$ is not only right-complete in A^* but it is also a maximal prefix code. Similarly, we can prove that $H_1\cup H_L\cup\bar{D}_X\cup Y$ is a maximal suffix code as well. By noting that

$$\begin{aligned}
\pi(H_1\cup H_R\cup\bar{D}_X\cup Y) &= \pi(H_1)+\pi(H_R)+(1-\beta)+(\beta-\alpha)=1,\\
\pi(H_1\cup H_L\cup\bar{D}_X\cup Y) &= \pi(H_1)+\pi(H_L)+(1-\beta)+(\beta-\alpha)=1
\end{aligned}$$

and

$$\pi(D_X)=\pi(H_1\cup H_2\cup H_L\cup H_R)=m+m'+2\pi(H_L)=\beta,$$

we see immediately that $\pi(H_L)=\pi(H_R)=\alpha-m=(\beta-m-m')/2$. The proof is completed. □

Lemma 3.5. *$\bar{D}_X\cup E_X\cup H_2\cup H_L(Y\cup\bar{D}_X\cup H_1)^*H_R$ is a maximal bifix code. If we let $W'=H_2\cup H_L(Y\cup\bar{D}_X\cup H)^*H_R$. Then W' is a maximal bifix code in B_X satisfying $W'=S(X^{d'})((S(X^{d'}))^{-1}W'(P(X^d))^{-1})P(X^d)$.*

Proof. It is easy to see that $Y \cup \bar{D}_X \cup H_1$ is a bifix code. Since $H_L \subseteq G_X$, there does not exist any suffix relation between H_L and $\bar{D}_X$. Observe that each word in H_L has a suffix in E_X, there does not exist any suffix relation between H_L and Y. Moreover, because H_L and H_1 are the parts of G_X, clearly, there does not exist any suffix relation between these two parts, H_L and H_1. Thus, there does not exist any suffix relation between H_L and $Y \cup \bar{D}_X \cup H_1$. Similarly, there does not exist any prefix relation between H_R and $Y \cup \bar{D}_X \cup H_1$. This shows that $H_L(Y \cup \bar{D}_X \cup H_1)^* H_R$ must be a bifix code. Among the codes $\bar{D}_X, E_X, H_2$ and $H_L(Y \cup \bar{D}_X \cup H_1)^* H_R$, it is clear that there does not exist any prefix relations and suffix relation between each of them, consequently, $\bar{D}_X \cup E_X \cup H_2 \cup H_L(Y \cup \bar{D}_X \cup H_1)^* H_R$ is a bifix code. In order to show that the above code is a maximal bifix code, we can count their measure. Because

$$
\begin{aligned}
&\pi(\bar{D}_X \cup E_X \cup H_2 \cup H_L(Y \cup \bar{D}_X \cup H_1)^* H_R \\
&\quad = (1-\beta) + \alpha + m' + (\alpha - m)^2 \left(\sum_{i=0}^{\infty} ((\beta - \alpha) + (1-\beta) + m)^i \right) \\
&\quad = (1-\beta) + \alpha + m' + (\alpha - m) \\
&\quad = (1-\beta) + \alpha + (\beta - m - 2(\alpha - m)) + \alpha - m \\
&\quad = 1.
\end{aligned}
$$

We see immediately that $\bar{D}_X \cup E_X \cup H_2 \cup H_L(Y \cup \bar{D}_X \cup H_1)^* H_R$ is a maximal bifix code. Hence, by Lemma 3.2, we conclude that W' is a maximal bifix code in B_X. Thus by the construction of W', we can easily check that $W' = S(X^{d'})((S(X^{d'}))^{-1} W' (P(X^d))^{-1}) P(X^d)$. The proof is completed. □

Theorem 3.6. *Let $X \subseteq A^+$ be a rational (d, d')-bifix code. Then there exists $W \subseteq A^* - X$ such that $X \cup W$ is a rational maximal (d, d')-bifix code over A.*

Proof. By Lemma 3.5, we see immediately that W' is a maximal bifix code in B_X and therefore, by Lemma 3.5 again, we can write

$$
W' = S(X^{d'})((S(X^{d'}))^{-1} W' (P(X^d))^{-1}) P(X^d).
$$

Now, let $W = (S(X^{d'}))^{-1} W' (P(X^d))^{-1}$. Then, by Theorem 2.11, we see immediately that $X \cup W$ is rational maximal (d, d')-bifix code over A. This proves our Theorem 3.6. □

4. Noetherian (d, d')-bifix codes

Bruyere and Perrin have recently extended the method of completion for codes to *Noetherian codes* [3]. To further extend their results, we now give here a method of completion for the Noetherian (d, d')-bifix codes. We start with the following definition.

Definition 4.1. A language $L \subseteq A^*$ is called a Noetherian language if $A^- L$ and LA^- satisfy the chain conditions in the prefix order and suffix order, respectively.

Proposition 4.1. *Let L_1 and L_2 be Noetherian languages. Then the following statements hold*:
(1) *$L_1 \cup L_2$ and $L_1 L_2$ are Noetherian languages.*
(2) *If $L \subseteq L_1$, then L is a Noetherian language.*

Proof. We only prove that $L_1 L_2$ is a Noetherian language because the other statements are trivial. If $A^-(L_1 L_2)$ has an infinite chain

$$s_1 \overset{P}{<} s_2 \overset{P}{<} \cdots \overset{P}{<} s_n \overset{P}{<} \cdots,$$

which is ordered by the prefix order, then there are at most finite number of words s_i's belonging to $A^- L_2$, i.e., there are infinite number of words s_i's belonging to $(A^- L_1)L_2$ since L_2 is a Noetherian language. If $s_i \in (A^- L_1)L_2$, then we can express s_i by $s_i = r_i t_i$, where $r_i \in A^- L_1$ and $t_i \in L_2$. Since L_1 is a Noetherian language, there are infinite number of words r_i which are all equal and hence their corresponding elements t_i's form an infinite chain under the prefix order. However, this contradicts our assumption that L_2 must be a Noetherian language. Therefore, $L_1 L_2$ is a Noetherian language. $\square$

The above proposition shows that the Noetherian languages are all closed under the operations of union, concatenations and subsets.

Proposition 4.2. *If X is a Noetherian language, then X is thin.*

Proof. Let X be a Noetherian language. Consider the following chain:

$$x_1 \overset{P}{<} x_2 \overset{P}{<} \cdots \overset{P}{<} x_n$$

in $A^- X$ under the prefix order. If X is dense, then for any word y satisfying $x_n \overset{P}{<} y$, we see that ay, where $a \in A$, is a factor of a word in X. Thereby, there exists a word $u \in A^*$ such that $ayu \in (A^*)^{-1}X$ and moreover, we have $yu \in A^- X$. This implies that the chain in $A^- X$ under the prefix order is, of course, not bounded, and hence its length is not finite. However, this clearly contradicts to our assumption that X is a Noetherian code. Hence, X must be a thin language. $\square$

For finishing the completion of Noetherian (d, d')-bifix codes, we still need the following lemmas:

Lemma 4.3. *Let X, Y and Z be the Noetherian bifix codes such that $X \cup Y$ and $Y \cup Z$ are bifix codes and $X \cap Y = \emptyset$, $Y \cap Z = \emptyset$. Then XY^*Z is a Noetherian bifix code.*

Proof. It is easy to see that XY^*Z is a bifix code. We now show that XY^*Z is a Noetherian code. Assume that

$$p_1 \overset{P}{<} p_2 \overset{P}{<} \cdots \overset{P}{<} p_n \overset{P}{<} \cdots$$

under the prefix order is an infinite chain in $A^{-}(XY^{*}Z)$. Then the following cases arise:

(1) There are infinite number of words p_i's in $(A^{-}X)Y^{*}Z$;
(2) There are infinite number of words p_i's in $(A^{-}Y^{*})Z$;
(3) There are infinite number of words p_i's in $A^{-}Z$.

For case (1), we can express the words p_i's by $p_i = s_i t_i$, where $s_i \in A^{-}X$ and $t_i \in Y^{*}Z$. Since $Y^{*}Z$ is a prefix code, we have $s_i \neq s_j$ if $i \neq j$. Thus, s_i's form an infinite chain under the prefix order. This contradicts X is a Noetherian code.

For case (2), we can express the words p_i's by $p_i = u_i v_i$, where $u_i \in A^{-}Y^{*}$ and $v_i \in Z$. Moreover, u_i can be expressed by $u_i = u_{i1}u_{i2}$, where $u_{i1} \in (A^{*})^{-1}Y$ and $u_{i2} \in Y^{*}$. Again, since $Y^{*}Z$ is a prefix code, all the u_{i1}'s form an infinite chain under the prefix order. This contradicts Y is a Noetherian code.

Case (3) cannot occur because it contradicts Z being a Noetherian code.

By summarizing the above facts, we can always find the lengths of chains in $A^{-}(XY^{*}Z)$ because these chains under the prefix order are all bounded. Similarly, we can also find the lengths of the chains in $(XY^{*}Z)A^{-}$ under the suffix order because these kind of chains are also bounded. Therefore, $XY^{*}Z$ must be a Noetherian code. □

Theorem 4.4. *Any Noetherian (d, d')-bifix code is included in a maximal one.*

Proof. In Section 2, we have already seen that all the chains are bounded for thin codes. Since all the Noetherian code are thin by Proposition 4.2, these results also hold for Noetherian codes. By Proposition 4.1, we see that D_X, E_X are Noetherian codes. Also, by [3], the codes $\bar{D}_X$ and Y constructed by the method given in [3] are Noetherian codes. To construct G_X, we can use the chain conditions of E_X because these conditions still hold under our new hypothesis. Now, by Lemma 4.3, we see that G_X is a Noetherian code. Therefore the final maximal (d, d')-bifix code $X \cup W$ is a Noetherian code. □

In closing this paper, we remark here that the class of rational bifix codes is contained in the class of Noetherian bifix code; however, the similar relation does not hold for the rational (d, d')-bifix codes and Noetherian (d, d')-bifix codes. For instance, we can see that $X = ab^{*}$ is a rational $(1, 0)$-bifix code but X is clearly not a Noetherian $(1, 0)$-bifix code.

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